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A VARIATIONAL APPROACH IN THEORIES OF FILTRATIONAL CONSOLIDATION AND TWO-PHASE FILTRATION†

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A general approach to the construction of dual variational principles for the linear problems of filtrational consolidation and two-phase filtration of an incompressible fluid is proposed. If the porosity and saturation are known, variational principles enable one to determine the displacement and stress fields in the solid phase as well as the pressure and velocity fields in the liquid phase. The variational principles can be derived from variational problems, the solution of which is equivalent to ensuring that the defining relations between the strains and stresses as well as the rate of filtration and the pressure gradient are satisfied. Using variational principles, it is shown that consolidation and filtration problems can be split into problems characterizing the behaviour of the individual phases. Thus the construction of the variational principles can be reduced to a certain connection scheme between the variational principles for the solid and liquid phases.

1. THE VARIATIONAL APPROACH

CONSIDER a dissipative process with volume dissipation, which can be written in the form

$$\Sigma = \mathbf{XY} = \mathbf{X}_1\mathbf{Y}_1 + \ldots + \mathbf{X}_n\mathbf{Y}_n$$

where $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ are the generalized forces and $\mathbf{Y} = (\mathbf{Y}_1, \ldots, \mathbf{Y}_n)$ are the generalized velocities. According to the normal dissipation hypothesis [1, 2], for any real thermodynamic process, there exists a dissipation potential $\varphi(\mathbf{Y})$ such that

$$\mathbf{X} \in \partial\varphi(\mathbf{Y}) \tag{1.1}$$

where $\varphi(\mathbf{Y})$ is the convex lower semicontinuous characteristic functional, $(\mathbf{X})$ is a subgradient of $\varphi(\mathbf{Y})$ at $\mathbf{Y}$, and $\partial\varphi(\mathbf{Y})$ is the set of all subgradients of $\varphi(\mathbf{Y})$ at $\mathbf{Y}$, which consists of the single element $\operatorname{grad}\varphi(\mathbf{Y})$ in the case of smooth $\varphi(\mathbf{Y})$. A vector $\mathbf{X}$ corresponds to the given vector $\mathbf{Y}$ if and only if $\mathbf{X}$ is a subgradient of $\varphi(\mathbf{Y})$ at $\mathbf{Y}$. Formula (1.1) implies the inverse relation [1, 2]

$$\mathbf{Y} \in \partial\varphi^*(\mathbf{X}) \tag{1.2}$$

where the adjoint dissipation potential $\varphi^*(\mathbf{X})$ is the convex lower semicontinuous characteristic functional connected with $\varphi(\mathbf{Y})$ by the Young–Fenchel transformation [3]

$$\varphi^*(\mathbf{X}) = \sup_{\mathbf{Y}}[\mathbf{XY} - \varphi(\mathbf{Y})]$$

For smooth convex potential $\varphi(\mathbf{Y})$ and $\varphi^*(\mathbf{X})$, we can write

$$\mathbf{X} = \mathbf{grad}\ \varphi(\mathbf{Y}),\ \mathbf{Y} = \mathbf{grad}\ \varphi^*(\mathbf{X})$$

instead of (1.1) and (1.2).

It is customary to assume that the laws (1.1) and (1.2), which enable one to state the relations

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between X and Y, determine the dissipative mechanism [2]. If there are two dissipative mechanisms with potentials $\Psi_1(\mathbf{Y})$, and $\Psi_2(\mathbf{Y})$, then one can define a new mechanism by means of the potential

$$\varphi(\mathbf{Y}) = \Psi_1(\mathbf{Y}) + \Psi_2(\mathbf{Y})$$

In the general case $\varphi^*(\mathbf{X})$ is not equal to the sum of the potentials $\Phi_1(\mathbf{X})$ and $\Phi_2(\mathbf{X})$ adjoint to $\Psi_1(\mathbf{Y})$, and $\Psi_2(\mathbf{Y})$, respectively. The dissipative mechanisms $\Psi_1(\mathbf{Y})$ and $\Psi_2(\mathbf{Y})$ are assumed to be unconnected [2] if $\Psi_2(\mathbf{Y})$ is independent of any of those variables $\mathbf{Y}_\alpha$ on which $\Psi_1(\mathbf{Y})$ depends. In this case we can write

$$\varphi(\mathbf{Y}) = \varphi(\mathbf{Y}_1, \mathbf{Y}_2) = \Psi_1(\mathbf{Y}_1) + \Psi_2(\mathbf{Y}_2), \ \mathbf{Y}_1 \in E_p, \ \mathbf{Y}_2 \in E_q$$

$$\varphi^*(\mathbf{X}) = \varphi^*(\mathbf{X}_1, \mathbf{X}_2) = \Phi_1(\mathbf{X}_1) + \Phi_2(\mathbf{X}_2), \ \mathbf{X}_1 \in E_p, \ \mathbf{X}_2 \in E_q$$

where E_p and E_q are two mutually complementary subspaces of dimensions p and q such that $p+q=n$.

The following assertions are equivalent [1]:

(a) $\mathbf{X}' \in \partial\varphi(\mathbf{Y}')$,

(b) $\varphi(\mathbf{Y}) - \mathbf{X}'\mathbf{Y}$ reaches a minimum with respect to Y at $\mathbf{Y} = \mathbf{Y}'$,

(a*) $\mathbf{Y}' \in \partial\varphi^*(\mathbf{X}')$,

(b*) $\varphi^*(\mathbf{X}) - \mathbf{X}\mathbf{Y}'$ reaches a minimum with respect to X at $\mathbf{X} = \mathbf{X}$.

Hence it follows that, for the dual process $(\mathbf{X}^\circ\mathbf{Y}^\circ)$ in Ω, the quantity $\mathbf{Y}^\circ$ corresponding to $\mathbf{X}^\circ$ can be determined by means of the solution of the problem

$$\inf_{\mathbf{Y}} B_1^\circ(\mathbf{Y}) = \inf_{\mathbf{Y}} \int_\Omega [\varphi(\mathbf{Y}) - \mathbf{X}^\circ\mathbf{Y}]\, d\Omega \tag{1.3}$$

In this formulation the definition of $\mathbf{Y}^\circ$ is trivial, since it is necessary to specify the forces $(\mathbf{X}_1^\circ, \ldots, \mathbf{X}_n^\circ)$ in the entire domain Ω. The problem arises from transforming the integral $\int_\Omega \mathbf{X}^\circ\mathbf{Y}\, d\Omega$ using the given constraints into an integral over the boundary Γ of Ω. In this case, to determine $\mathbf{Y}^\circ$, it suffices to know the forces $(\mathbf{X}_1^\circ, \ldots, \mathbf{X}_n^\circ)$ on Γ. A similar problem also arises when determining $\mathbf{X}^\circ$

$$\inf_{\mathbf{X}} B_2^\circ(\mathbf{X}) = \inf_{\mathbf{X}} \int_\Omega [\varphi^*(\mathbf{X}) - \mathbf{X}\mathbf{Y}^\circ]\, d\Omega \tag{1.4}$$

The normal dissipation hypothesis of the form (1.3), (1.4) is the principle of least energy dissipation [4, 5] if a wider class of functions $\varphi(\mathbf{Y})$ and $\varphi^*(\mathbf{X})$ is taken into account.

For the dissipation potential $\varphi(\mathbf{Y})$, the Young–Fenchel transformation with respect to some of the variables $\mathbf{Y}_\alpha$ determines the partially adjoint potentials. For a system with two independent dissipative mechanisms $\Psi_1(\mathbf{Y}_1)$ and $\Psi_2(\mathbf{Y}_2)$, we can apply a transformation with respect to the variables involved in $\mathbf{Y}_2$ to obtain the partially adjoint potential

$$\varphi(\mathbf{Y}_1, \mathbf{X}_2) = \sup_{\mathbf{Y}_2} [\mathbf{X}_2\mathbf{Y}_2 - \varphi(\mathbf{Y}_1, \mathbf{Y}_2)] = \Phi_2(\mathbf{X}_2) - \Psi_1(\mathbf{Y}_1)$$

The variational problem for constructing the variational principle in $\mathbf{Y}_1$ and $\mathbf{X}_2$ will then have the form

$$\inf_{\mathbf{Y}_1} \sup_{\mathbf{X}_2} B_3^\circ(\mathbf{Y}_1, \mathbf{X}_2) = \inf_{\mathbf{Y}_1} \sup_{\mathbf{X}_2} \int_\Omega [\Psi_1(\mathbf{Y}_1) - \mathbf{X}_1^\circ\mathbf{Y}_1 - \Phi_2(\mathbf{X}_2) + \mathbf{X}_2\mathbf{Y}_2^\circ)]\, d\Omega \tag{1.5}$$

Instead of using problems (1.3), (1.4) and (1.5) to construct the variational principles, one can start from the variations

$$\delta B_1(\mathbf{Y}) = \int_\Omega [\delta\varphi(\mathbf{Y}) - \mathbf{X}\delta\mathbf{Y}]\, d\Omega, \quad \delta B_2(\mathbf{X}) = \int_\Omega [\delta\varphi^*(\mathbf{X}) - \mathbf{Y}\delta\mathbf{X}]\, d\Omega$$

$$\delta B_3(\mathbf{Y}_1, \mathbf{X}_2) = \int_\Omega [\delta\Psi_1(\mathbf{Y}_1) - \mathbf{X}_1\delta\mathbf{Y}_1 - \delta\Phi_2(\mathbf{X}_2) + \mathbf{Y}_2\delta\mathbf{X}_2]\, d\Omega$$

which, on equating to zero, give

$$\delta B_1(\mathbf{Y}) = 0,\ \delta B_2(\mathbf{X}) = 0,\ \delta B_3(\mathbf{Y}_1, \mathbf{X}_2) = 0 \tag{1.6}$$

which is equivalent to satisfying the defining relations between **Y** and **X**. Here the problem consists in reducing the variations $\delta B_1(\mathbf{Y})$, $\delta B_2(\mathbf{X})$ and $\delta B_3(\mathbf{Y}_1, \mathbf{X}_2)$ to the variations of some functionals.

Variational problems similar to (1.3)–(1.6) can be written down for any convex potential connecting arbitrary dual variables **X** and **Y** by means of (1.1) or (1.2), and they can be used to construct variational principles.

2. TWO-PHASE FILTRATION OF AN INCOMPRESSIBLE FLUID

We shall write down the equations of continuity for the phases, the relation between the pressures in the phases, and an expression for entropy production [6] due to the motion of the liquid phases with respect to the solid phase for $T_1 \approx T_2 \simeq \text{const}$ in the energy representation:

$$(sm)_{,t} + \operatorname{div}(sm\mathbf{v}_1) = 0,\quad ((1-s)\,m)_{,t} + \operatorname{div}((1-s)\,m\mathbf{v}_2) = 0$$
$$p_1 - p_2 = p_c(s),\ \Sigma = -\nabla p_1 \mathbf{q}_1 - \nabla p_2 \mathbf{q}_2$$

Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of motion of the phases, s is the saturation of the first phase, m is the porosity, $\mathbf{q}_1 = sm\mathbf{v}_1$ and $\mathbf{q}_2 = (1-s)m\mathbf{v}_2$ are the filtration velocities of the phases and T_1, T_2 are the absolute temperatures of the phases.

To close the equations, we can use the normal dissipation hypothesis, according to which there exist convex dissipation potentials

$$\varphi(\mathbf{Y}_1, \mathbf{Y}_2),\ \varphi^*(\mathbf{X}_1, \mathbf{X}_2),\ \text{where } \mathbf{X}_1 = -\nabla p_1, \mathbf{X}_2 = -\nabla p_2, \mathbf{Y}_1 = \mathbf{q}_1, \mathbf{Y}_2 = \mathbf{q}_2.$$

We will assume that the dissipation process can be represented by two unconnected dissipative mechanisms. Then

$$\varphi(\mathbf{Y}_1, \mathbf{Y}_2) = \Psi_1(\mathbf{q}_1) + \Psi_2(\mathbf{q}_2),\ \varphi^*(\mathbf{X}_1, \mathbf{X}_2) = \Phi_1(\nabla p_1) + \Phi_2(\nabla p_2) \tag{2.1}$$

where Ψ_i and Φ_i are the dissipative and adjoint dissipative potentials for the liquid phase [7].

Taking (2.1) into account, we can write down the system of equations of two-phase filtration in the form

$$-\mathbf{q}_i \in \partial\Phi_i(\nabla p_i) \ \text{ or } \ -\nabla p_i \in \partial\Psi_i(\mathbf{q}_i),\ i = 1, 2 \tag{2.2}$$

$$p_1 - p_2 = p_c(s) \tag{2.3}$$

$$\operatorname{div}\mathbf{q}_1 + \operatorname{div}\mathbf{q}_2 = 0 \tag{2.4}$$

$$\operatorname{div}\mathbf{q}_1 = -ms_{,t} \ \text{ or } \ \operatorname{div}\mathbf{q}_2 = ms_{,t} \tag{2.5}$$

In the case of smooth potentials the filtration laws (2.2) for the phases have the form

$$-q_{1i} = \partial\Phi_1(\nabla p_1)/\partial p_{1,i} \ \text{ or } \ -p_{1,i} = \partial\Psi_1(\mathbf{q}_1)/\partial q_{1i}$$
$$-q_{2i} = \partial\Phi_2(\nabla p_2)/\partial p_{2,i} \ \text{ or } \ -p_{2,i} = \partial\Psi_2(\mathbf{q}_2)/\partial q_{2i}$$

where q_{1i}, q_{2i} and $p_{1,i}$, $p_{2,i}$ are the components of $\mathbf{q}_1$, $\mathbf{q}_2$ and ∇p_1, ∇p_2, respectively.

In particular, the dissipation potentials $\Phi_1(\nabla p_1)$, and $\Phi_2(\nabla p_2)$ can have the following expressions [7]:

$$\Phi_1(\nabla p_1) = \int_0^{|\nabla p_1|} \varphi_1(\alpha, s)\,d\alpha,\quad \Phi_2(\nabla p_2) = \int_0^{|\nabla p_2|} \varphi_2(\alpha, s)\,d\alpha$$

where the convexity of $\Phi_1(\nabla p_1)$ and $\Phi_2(\nabla p_2)$ is ensured by the properties of $\varphi_1(\alpha, s)$ and $\varphi_2(\alpha, s)$. For the linear filtration laws

$$\mathbf{q}_i = -k\,(f_i(s)/\mu_i)\,\nabla p_i \tag{2.6}$$

the functions $\varphi_1(\alpha, s)$ and $\varphi_2(\alpha, s)$ have the form

$$\varphi_i(\alpha, s) = k(f_i(s)/\mu_i)\alpha$$

where k is the absolute permeability, $f_1(s)$ $(i = 1, 2)$ are the relative permeabilities of the phases and μ_i $(i = 1, 2)$ are the viscosities. The following boundary conditions are given for the solution of system (2.2)–(2.5):

$$p|_{\Gamma_p} = p^\circ \tag{2.7}$$

$$q_n|_{\Gamma_q} = q_n^\circ \tag{2.8}$$

Here $\Gamma = \Gamma_q + \Gamma_p$ is the boundary of the domain Ω of the solution of the problem, $q_n = q_{1n} + q_{2n}$ is the normal component of the total filtration velocity $p = lp_1 + (1-l)p_2$ with $l = l(s)$ such that $p = p_1$ and $p = p_2$ for $l = 1$ and $l = 0$, respectively, p is the mean pressure for $l(s) = s$ or $l(s) = F(s)$ in the Buckley–Leverett function.

Equations (2.2) play a fundamental role in the construction of variational principles. Equations (2.3), (2.4) and the boundary conditions (2.7), (2.8) are used as constraints in the construction. We shall construct a variational principle in terms of the velocities. From (1.3) we have

$$B_1^\circ(\mathbf{q}_1, \mathbf{q}_2) = \int_\Omega \{[\Psi_1(\mathbf{q}_1) + \Psi_2(\mathbf{q}_2)] + (\mathbf{q}_1\nabla p_1^\circ + \mathbf{q}_2\nabla p_2^\circ)\}\,d\Omega \tag{2.9}$$

Transforming the right-hand side of (2.9), taking the constraints (2.3), (2.4), and (2.8) into account, we obtain the functional

$$I_1(\mathbf{q}_1, \mathbf{q}_2) = \int_\Omega [\Psi_1(\mathbf{q}_1) + \mathbf{q}_1\nabla((1-l)p_c)]\,d\Omega + \\ + \int_\Omega [\Psi_2(\mathbf{q}_2) - \mathbf{q}_2\nabla(lp_c)]\,d\Omega + \int_{\Gamma_p} q_n p^\circ d\Gamma \tag{2.10}$$

It can be verified directly that a minimum of (2.10) subject to the constraints (2.4) and (2.8) is attained for the real velocity field $\mathbf{q}_1$, $\mathbf{q}_2$. By analogy, from (1.4) we can construct the functional

$$I_2(p) = \int_\Omega [\Phi_1(\nabla(p + (1-l)p_c)) + \Phi_2(\nabla[(p - lp_c))]\,d\Omega + \int_{\Gamma_q} q_n^\circ p\,d\Gamma$$

whose minimum subject to the constraint (2.7) is attained for the real pressure field p.

Applying duality methods [3], we can find that the dual to the variational problem

$$\inf_{\mathbf{q}_1, \mathbf{q}_2 \in (2.4), (2.8)} I_1(\mathbf{q}_1, \mathbf{q}_2) \tag{2.11}$$

is the problem for the maximum of the functional $[-I_2(p)]$, i.e.

$$\inf_{\mathbf{q}_1, \mathbf{q}_2 \in (2.4), (2.8)} I_4(\mathbf{q}_1, \mathbf{q}_2) = \sup_{p \in (2.7)} [-I_2(p)]$$

We write down the boundary conditions

$$q_{1n}|_{\Gamma_q} = q_{1n}^\circ, \quad q_{2n}|_{\Gamma_q} = q_{2n}^\circ \tag{2.12}$$

and we transfer from (2.11) to the dual problem with respect to one of the variables. We obtain two minimax problems for I_3 and I_4 such that

$$\sup_{p \in (2.7)} \inf_{\mathbf{q}_1 \in (2.12)} I_3(\mathbf{q}_1, p) = \inf_{\mathbf{q}_1, \mathbf{q}_2 \in (2.4), (2.8)} I_1(\mathbf{q}_1, \mathbf{q}_2)$$

$$\sup_{p \in (2.7)} \inf_{\mathbf{q}_2 \in (2.12)} I_1(p, \mathbf{q}_2) = \inf_{\mathbf{q}_1 \mathbf{q}_2 \in (2.4), (2.8)} I_1(\mathbf{q}_1, \mathbf{q}_2)$$

$$I_3(\mathbf{q}_1, p) = \int_\Omega [\Psi_1(\mathbf{q}_1) + \mathbf{q}_1\nabla((1-l)p_c)]\,d\Omega + \int_{\Gamma_p} p^\circ q_{1n}\,d\Gamma -$$

$$-\int_{\Omega}\Phi_2(\nabla(p-lp_c))\,d\Omega-\int_{\Gamma_q}p\overset{\circ}{q}_{2n}\,d\Gamma-\int_{\Omega}p\,\mathrm{div}\,\mathbf{q}_1\,d\Omega$$

$$I_4(p,\mathbf{q}_2)=-\int_{\Omega}\Phi_1(\nabla(p+(1-l)\,p_c))\,d\Omega-\int_{\Gamma_q}p\overset{\circ}{q}_{1n}\,d\Gamma+$$

$$+\int_{\Omega}[\Psi_2(\mathbf{q}_2)-\mathbf{q}_2\nabla(lp_c)]\,d\Omega+\int_{\Gamma_p}p^\circ q_{2n}\,d\Gamma-\int_{\Omega}p\,\mathrm{div}\,\mathbf{q}_2\,d\Omega$$

In the case of the linear filtration laws (2.6), from the functional (2.10), in which we set $l(s)=F(s)$ and we express $\mathbf{q}_1$ and $\mathbf{q}_2$ in terms of the total velocity $\mathbf{q}=\mathbf{q}_1+\mathbf{q}_2$, we get

$$I_1(\mathbf{q})=\int_{\Omega}\left[\frac{1}{2}\frac{\mu_1}{k\varphi(s)}|\mathbf{q}|^2+(p_c\nabla F(s))\,\mathbf{q}\right]d\Omega+\int_{\Gamma_p}q_np^\circ d\Gamma \tag{2.13}$$

$$F(s)=f_1(s)/\varphi(s),\quad \varphi(s)=f_1(s)+(\mu_1/\mu_2)\,f_2(s)$$

where $\varphi(s)$ is the relative mobility. After some transformations, the functional (2.13) can be written in the form

$$I_1(\mathbf{q})=\int_{\Omega}\frac{1}{2}\frac{\mu_1}{k\varphi(s)}|\mathbf{q}|^2\,d\Omega+\int_{\Gamma_p}q_np_*^\circ d\Gamma \tag{2.14}$$

$$p_*=p_1F(s)+p_2(1-F(s))+\int_s^1 p_c(s)\,dF(s)$$

Under the constraints

$$\mathrm{div}\,\mathbf{q}=0,\quad q_n|_{\Gamma_q}=q_n^\circ$$

the minimum of (2.13), (2.14) is attained for the real total velocity field $\mathbf{q}$.

3. FILTRATIONAL CONSOLIDATION

We consider the variational approach for problems with a viscoplastic framework. We write down the equations of continuity for the phases, the equilibrium equation, and the expression for entropy production [6] due to the work $\mathbf{q}\nabla p$ in the liquid phase and $\sigma_{ij}{}^f e_{ij}$ in the solid phase for $T_1\approx T_2\approx$const in the energy representation:

$$-m_{,t}+\mathrm{div}\,((1-m)\,\mathbf{u}^{\boldsymbol{\cdot}})=0,\quad m_{,t}+\mathrm{div}\,(m\mathbf{v})=0$$

$$\sigma^f_{ij,j}-p_{,i}=0,\quad \Sigma=\sigma_{ij}{}^f e_{ij}-\mathbf{q}\nabla p$$

Here $\mathbf{q}=m(\mathbf{v}-\mathbf{u}^{\boldsymbol{\cdot}})$ is the filtration velocity, $\mathbf{v}$ is the velocity of motion of the liquid phase, $\mathbf{u}$ is the displacement vector of the solid phase, p is the porous pressure, σ^f is the effective stress tensor, $\mathbf{e}$ is the viscoplastic deformation velocity tensor and T_1 and T_2 are the absolute temperatures of the phases.

According to the normal dissipation hypothesis, there exist convex dissipation potentials $\varphi(\mathbf{Y}_1,\mathbf{Y}_2)$ and $\varphi^*(\mathbf{X}_1,\mathbf{X}_2)$, where $\mathbf{X}_1=\boldsymbol{\sigma}^f$, $\mathbf{X}_2=-\nabla p$, $\mathbf{Y}_1=\mathbf{e}$ and $\mathbf{Y}_2=\mathbf{q}$. Assuming that the process of dissipation can be represented by two unconnected dissipative mechanisms,

$$\varphi(\mathbf{Y}_1,\mathbf{Y}_2)=\Psi_e(\mathbf{e})+\Psi_q(\mathbf{q}),\quad \varphi^*(\mathbf{X}_1,\mathbf{X}_2)=\Phi_\sigma(\boldsymbol{\sigma}^f)+\Phi_p(\nabla p)$$

we can write down the system of filtration consolidation in the form

$$-\mathbf{q}\in\partial\Phi_p(\nabla p)\quad\text{or}\quad -\nabla p\in\partial\Psi_q(\mathbf{q}) \tag{3.1}$$

$$\mathbf{e}\in\partial\Phi_\sigma(\boldsymbol{\sigma}^f)\quad\text{or}\quad \boldsymbol{\sigma}^f\in\partial\Psi_e(\mathbf{e}) \tag{3.2}$$

$$\sigma^f_{ij,j}-p_{,i}=0 \tag{3.3}$$

$$\operatorname{div} \mathbf{q} + \operatorname{div} \mathbf{u}^{\cdot} = 0 \tag{3.4}$$

$$m_{,t} = \operatorname{div} (1 - m) \mathbf{u}^{\cdot} \tag{3.5}$$

where Ψ_q, Φ_p and Ψ_e, Φ_σ are the dissipative and adjoint dissipative potentials for the liquid phase [7] and the viscoplastic framework [8], respectively.

In the case of smooth potentials the laws (3.1) and (3.2) governing the behaviour of the phases have the form

$$-q_i = \partial\Phi_p(\nabla p)/\partial p_{,i} \quad \text{or} \quad -p_{,i} = \partial\Psi_q(\mathbf{q})/\partial q_i$$

$$e_{ij} = \partial\Phi_\sigma(\sigma^f)/\partial\sigma_{ij}{}^f \quad \text{or} \quad \sigma_{ij}{}^f = \partial\Psi_e(\mathbf{e})/\partial e_{ij}$$

where e_{ij}, q_i and $\sigma_{ij}{}^f$, $p_{,i}$ are the components of $\mathbf{e}$, $\mathbf{q}$ and σ^f, ∇p, respectively.

In problems concerned with consolidation the variation of the porosity m is usually neglected and Eq. (3.5) is not used. The following boundary conditions are given for solving the system of equations (3.1)–(3.4):

$$p \,|_{\Gamma_p} = p^\circ, \quad q_n \,|_{\Gamma_q} = q_n{}^\circ \tag{3.6}$$

$$(\sigma_{ij}{}^f - p\delta_{ij}) n_j \,|_{\Gamma_\sigma} = \Pi_i{}^\circ, \quad u_i \,|_{\Gamma_u} = u_i{}^\circ \tag{3.7}$$

where $\Gamma = \Gamma_\sigma + \Gamma_u = \Gamma_p + \Gamma_q$ is the boundary of the domain of the solution of the problem.

Equations (3.1) and (3.2) are fundamental in the construction of the variational principles. Equations (3.3) and (3.4) and the boundary conditions (3.6) and (3.7) can be used as constraints. The structure of the system of equations (3.1)–(3.5) is similar to that of the system of equations (2.2)–(2.5) for two-phase filtration.

From (1.3), after suitable transformations, we can obtain the following functional in terms of the velocities:

$$I_1(\mathbf{u}^{\cdot}, \mathbf{q}) = \int_\Omega [\Psi_e(\mathbf{e}) + \Psi_q(\mathbf{q})] \, d\Omega - \int_{\Gamma_\sigma} \Pi_i{}^\circ u_i^{\cdot} \, d\Gamma + \int_{\Gamma_p} p^\circ q_n \, d\Gamma \tag{3.8}$$

It can be verified directly that the minimum of this functional subject to the constraints (3.4), (3.6), and (3.7) is attained for the real velocity field $\mathbf{u}$, $\mathbf{q}$. From (1.4) we obtain the functional

$$I_2(\sigma^f, p) = \int_\Omega [\Phi_\sigma(\sigma^f) + \Phi_p(\nabla p)] \, d\Omega - \int_{\Gamma_u} (\sigma_{ij}{}^f - p\delta_{ij}) n_j u_i^{\cdot\circ} \, d\Gamma + \int_{\Gamma_q} p q_n{}^\circ \, d\Gamma \tag{3.9}$$

the minimum of which subject to the constraints (3.3), (3.6), and (3.7) is attained for the real field of p, $\sigma_{ij}{}^f$. Applying duality methods [3], we find that the dual of the variational problem

$$\inf_{\mathbf{u}^{\cdot}, \mathbf{q} \in (3.4), (3.6), (3.7)} I_1(\mathbf{u}^{\cdot}, \mathbf{q}) \tag{3.10}$$

is the problem concerned with the maximum of the functional $[-I_2(\sigma^f, p)]$, i.e.

$$\inf_{\mathbf{u}^{\cdot}, \mathbf{q} \in (3.4), (3.6), (3.7)} I_1(\mathbf{u}^{\cdot}, \mathbf{q}) = \sup_{p, \sigma^f \in (3.3), (3.6), (3.7)} [-I_2(\sigma^f, p)]$$

Passing from (3.10) to the dual problem with respect to one of the variables, we can obtain two variational problems for I_3 and I_4 such that

$$\sup_{p \in (3.6)} \inf_{\mathbf{u}^{\cdot} \in (3.7)} I_3(\mathbf{u}^{\cdot}, p) = \inf_{\mathbf{u}^{\cdot}, \mathbf{q} \in (3.4), (3.6), (3.7)} I_1(\mathbf{u}^{\cdot}, \mathbf{q})$$

$$\sup_{p, \sigma^f \in (3.3), (3.7)} \inf_{\mathbf{q} \in (3.6)} I_4(\sigma^f, p, \mathbf{q}) = \inf_{\mathbf{u}^{\cdot}, \mathbf{q} \in (3.4), (3.6), (3.7)} I_1(\mathbf{u}^{\cdot}, \mathbf{q})$$

$$I_3(\mathbf{u}^{\cdot}, p) = \int_\Omega [\Psi_e(\mathbf{e}) - \Phi_p(\nabla p)] \, d\Omega - \int_{\Gamma_\sigma} \Pi_i{}^\circ u_i \, d\Gamma - \int_{\Gamma_q} p q_n{}^\circ \, d\Gamma - \int_\Omega p \operatorname{div} \mathbf{u}^{\cdot} \, d\Omega$$

$$I_4(\sigma^f, p, \mathbf{q}) = \int_\Omega [\Phi_\sigma(\sigma^f) - \Phi(\mathbf{q})]\, d\Omega - \int_{\Gamma_u} (\sigma_{ij}{}^f - p\delta_{ij})\, n_j u^{\cdot\circ}\, d\Gamma -$$

$$- \int_{\Gamma_p} p^\circ q_n\, d\Gamma + \int_\Omega p \operatorname{div} \mathbf{q}\, d\Omega \tag{3.11}$$

For $\Gamma_u = \Gamma$, $\Gamma_q = \Gamma$, the functional (3.8) has the form

$$I_1(\mathbf{u}^\cdot, \mathbf{q}) = \int_\Omega [\Psi_e(\mathbf{e}) + \Psi_q(\mathbf{q})]\, d\Omega \tag{3.12}$$

In this case, if $\Psi_e(\mathbf{e}) = D_e(\mathbf{e})$ and $\Psi_q(\mathbf{q}) = D_q(\mathbf{q})$, where D_e and D_q are dissipative functions, then the real process is determined by the minimum of the rate of energy dissipation.

For the problem of consolidation with elastic framework, the functional generalizing (3.12) has the form

$$I_1(\mathbf{u}, \mathbf{q}) = \int_\Omega \left[\frac{W_\varepsilon(\varepsilon(t)) - W_\varepsilon(\varepsilon(t) - \Delta t))}{\Delta t} + \Psi_q(\mathbf{q}) \right] d\Omega \tag{3.13}$$

where $W_\varepsilon(\varepsilon)$ is the convex elastic potential, $\sigma^f \in \partial W_\varepsilon(\varepsilon)$, and ε is the small-strain tensor. For $\Psi_q(\mathbf{q}) = D_q(\mathbf{q})$, the functional (3.13) approximately characterizes the sum of the energy storage and dissipation rates. In the general case it has the form

$$I_1(\mathbf{u}, \mathbf{q}) = \int_\Omega \left[\frac{W_\varepsilon(\varepsilon(t)) - W_\varepsilon(\varepsilon(t - \Delta t))}{\Delta t} + \Psi_q(\mathbf{q}) \right] d\Omega -$$

$$- \int_{\Gamma_\sigma} \Pi_i{}^\circ \frac{u_i(t) - u_i(t - \Delta t)}{\Delta t}\, d\Gamma + \int_{\Gamma_p} p^\circ q_n\, d\Gamma \tag{3.14}$$

The functionals dual to (3.14) are similar to (3.9) and (3.11).

We consider the functionals I_1 in the minimization problems (3.8) and (3.14). The variables $\mathbf{u}^\cdot$, $\mathbf{q}$ in (3.8) are connected by (3.4). If the quantities

$$\operatorname{div} \mathbf{u}^\cdot = \chi(\mathbf{r}, t), \ \operatorname{div} \mathbf{q} = -\chi(\mathbf{r}, t), \ \mathbf{r} \in \Omega \tag{3.15}$$

were known, then one could solve the problem of minimizing the functional $I_1(\mathbf{u}^\cdot, \mathbf{q})$ with respect to $\mathbf{u}^\cdot$, $\mathbf{q}$ and separately consider the problems

$$\inf J_1(\mathbf{u}^\cdot) \text{ for } \operatorname{div} \mathbf{u}^\cdot = \chi(\mathbf{r}, t) \tag{3.16}$$

and

$$\inf J_2(\mathbf{q}) \text{ for } \operatorname{div} \mathbf{q} = -\chi(\mathbf{r}, t) \tag{3.17}$$

i.e.

$$\inf_{\mathbf{u}^\cdot,\, \mathbf{q} \in (3.4),\, (3.6),\, (3.7)} I_1(\mathbf{u}^\cdot, \mathbf{q}) = \inf_{\mathbf{u}^\cdot \in (3.7),\, (3.15)} J_1(\mathbf{u}^\cdot) + \inf_{\mathbf{q} \in (3.6),\, (3.15)} J_2(\mathbf{q}) \tag{3.18}$$

By analogy, the problem of minimizing the functional (3.14) can be separated with respect to $\mathbf{u}$ and $\mathbf{q}$ and represented in a form similar to (3.18). The representations mentioned above split the problem of consolidation into two problems, one of which characterizes the process of strain, while the other one characterizes the process of filtration. In the problems

$$\sup_{\sigma^f,\, p \in (3.3),\, (3.7)} [-J_3(\sigma^f, p)], \quad \sup_{p \in (3.6)} [-J_4(p)]$$

dual to (3.16) and (3.17), respectively, the functionals $J_3(\sigma^f, p)$ and $J_4(p)$ have the form

$$J_3(\sigma^f, p) = \int_\Omega \Phi_\sigma(\sigma^f)\, d\Omega - \int_{\Gamma_u} (\sigma_{ij}{}^f - p\delta_{ij})\, n_j u^\circ\, d\Gamma - \int_\Omega p\chi(\mathbf{r}, t)\, d\Omega$$

$$J_4(p) = \int_\Omega \Phi_p(\nabla p)\, d\Omega + \int_{\Gamma_q} p q_n{}^\circ\, d\Gamma + \int_\Omega p\chi(\mathbf{r}, t)\, d\Omega$$

Introducting the Lagrange multiplier $\lambda = -p$, we write down the functionals

$$J_1'(\mathbf{u}^{\cdot}, p) = J_1(\mathbf{u}^{\cdot}) - \int_\Omega p(\operatorname{div}\mathbf{u}^{\cdot} - \chi(\mathbf{r}, t))\, d\Omega$$

$$J_2'(\mathbf{q}, p) = J_2(\mathbf{q}) - \int_\Omega p(\operatorname{div}\mathbf{q} + \chi(\mathbf{r}, t))\, d\Omega$$

Combining J_1', J_2', J_3, J_4 so as to eliminate $\chi(\mathbf{r}, t)$, we can obtain the functionals (3.9) and (3.11). A similar splitting also holds in problems concerned with two-phase filtration. Therefore, the problem of constructing the variational principles in the case of filtrational consolidation and two-phase filtration can be reduced to that of constructing the variational principles for the individual phases.

To construct the variational principles for consolidation problems with linear defining relations, one can use the method of [9, 10] based on the idea of constructing variational principles for equations with linear self-adjoint operators [11]. In this paper a non-linear form of the defining relations is assumed.

Using variational problems of the type (1.6), we shall present examples of constructing the variational principles.

Case 3. In the case of small strains, for the non-linear elastic behaviour $\sigma_{ij}^f = \partial W(\varepsilon)/\partial\varepsilon_{ij}$ of the solid phase and the non-linear behaviour $q_i = -\partial\Psi(\nabla p)/\nabla p_{,i}$ of the liquid phase, we write down the variation

$$\delta B_3(\varepsilon, p) = \int_\Omega \left[\delta\left(\frac{1}{2} C_{ijkl}\varepsilon_{kl} * \varepsilon_{ij}\right) - \sigma_{ij}^f * \delta\varepsilon_{ij}\right] d\Omega -$$

$$- \int_\Omega \left[\delta\left(1 * \frac{1}{2} k_{ij} p_{,j} * p_{,i}\right) + q_i * \delta p_{,i}\right] d\Omega$$

The variation vanishes if and only if the defining relations are satisfied. After some reduction we can obtain the variations $\delta I_3(\mathbf{u}, p)$, where [12]

$$I_3(\mathbf{u}, p) = \int_\Omega \left(\frac{1}{2} C_{ijkl}\varepsilon_{ijkl} * \varepsilon_{ij} - 1 * \frac{1}{2} k_{ij} p_{,i} * p_{,j}\right) d\Omega -$$

$$- \int_{\Gamma_\sigma} \Pi_i^{\circ} * u_i\, d\Gamma - \int_{\Gamma_q} 1 * q_n^{\circ} * p\, d\Gamma - \int_\Omega p * u_{i,i}\, d\Omega$$

Under the constraints (3.6) and (3.7), $\delta I_3(\mathbf{u}, p)$ is equal to zero for the real field $\mathbf{u}$, p.

By analogy, one can obtain the functionals I_1, I_2, I_4. The functionals I_1, I_2, I_3, I_4 for the linear viscoelastic framework $\sigma_{ij} = J_{ijkl} * \varepsilon_{kl}$ can be constructed in a similar way. The functional I_3 has been presented in [13].

Case 2. In the case of small strains, for a Kelvin–Voigt medium $\sigma_{ij}^f = \partial\Psi(\varepsilon, \varepsilon^{\cdot})/\partial\varepsilon_{ij}$ and the non-linear filtration law $\mathbf{q} = -f(|\nabla p|)\nabla p/|\nabla p|$, the functionals I_1, I_2, I_3, I_4 can be constructed by analogy with (3.8), (3.9), and (3.11). The functional

$$I_3(\mathbf{u}^{\cdot}, p) = \int_\Omega \Psi(\varepsilon, \varepsilon^{\cdot})\, d\Omega - \int_{\Gamma_\sigma} \Pi_i^{\circ} u_i^{\cdot}\, d\Gamma - \int_\Omega \int_0^{|\nabla p|} f(\alpha)\, d\alpha\, d\Omega - \int_{\Gamma_q} q_n^{\circ} p\, d\Gamma - \int_\Omega p u_{i,i}^{\cdot}\, d\Omega$$

has been obtained before.†

Case 3. In the case of small strains, for the non-linear elastic behaviour $\sigma_{ij}^f = \partial W(\varepsilon)/\partial\varepsilon_{ij}$ of the solid phase and the non-linear behaviour $q_i = -\partial\Psi(\nabla p)/\nabla p_{,i}$ of the liquid phase, we write down the variation

$$\delta B_3(\varepsilon, p) = \int_\Omega [\delta W(\varepsilon) - \sigma_{ij}^f \delta\varepsilon_{ij}]\, d\Omega - \int_0^t \int_\Omega [\Psi(\nabla p) + q_i \delta p_{,i}]\, d\Omega\, dt$$

†KOSTERIN A. V., The variational principle of filtrational consolidation. Kazan Univ., Kazan, 1986, deposited in VINITI 16.12.86, No. 8598–B.

which is equal to zero, if and only if the laws governing the behaviour of the phases are satisfied. After some reduction, we obtain the variation $\delta I_3(\mathbf{u}, p)$, where

$$I_3(\mathbf{u}, p) = \int_\Omega W(\varepsilon)\,d\Omega - \int_{\Gamma_\sigma} \Pi_i{}^\circ u_i\,d\Gamma - \int_0^t \left[\int_\Omega \Psi(\nabla p)\,d\Omega + \int_{\Gamma_q} q_n{}^\circ p\,d\Gamma - \int_\Omega p\,\mathrm{div}\,\mathbf{u}^\cdot\,d\Omega\right] dt$$

Under the constraints (3.6) and (3.7, the variation $\delta I_3(\mathbf{u}, p)$ vanishes for the solution of the problem.

Case 4. In the case of finite strains $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$, for the non-linear law $\sigma_{ij}{}^{f\cdot} = \partial W(\varepsilon^\cdot, \sigma^f)/\partial\varepsilon_{ij}{}^\cdot$ governing the behaviour of the solid phase and the linear filtration law $q_i = -k_{ij}p_{,j}$, we can write down the variation

$$\delta B_3(\varepsilon^\cdot, p^\cdot) = \int_\Omega [\delta W(\varepsilon^\cdot, \sigma^f) - \sigma_{ij}{}^{f\cdot}\delta\varepsilon_{ij}{}^\cdot]\,d\Omega - \int_0^t\int_\Omega \left[\delta\frac{1}{2}k_{ij}p^\cdot_{,i}p^\cdot_{,j} + q_i{}^\cdot\delta p_{,i}\right] d\Omega\,dt$$

which vanishes if and only if the laws governing the behaviour of the phases are satisfied. After some reduction, we obtain the variation $\delta I_3(u^\cdot, p)$, where [14]

$$I_3(\mathbf{u}^\cdot, p^\cdot) = \int_\Omega W(\varepsilon^\cdot, \sigma^f)\,d\Omega - \int_{\Gamma_\sigma} \Pi_i{}^{\cdot\circ}u_i{}^\cdot\,d\Gamma + \int_\Omega \frac{1}{2}(\sigma_{ij}{}^f - \delta_{ij}p)\,u^\cdot_{k,i}u^\cdot_{k,j}\,d\Omega -$$

$$-\int_0^t \left[\int_\Omega \frac{1}{2}k_{ij}p^\cdot_{,i}p^\cdot_{,j}\,d\Omega + \int_{\Gamma_q} q_n{}^{\cdot\circ}p^\cdot\,d\Gamma - \int_\Omega p^\cdot\varepsilon_{ii}{}^{\cdot\cdot}\,d\Omega\right] dt$$

For suitable restrictions [14] for the variables $u^\cdot, p^\cdot$, which are subject to variations, $\delta I_3(u^\cdot, p^\cdot)$ vanishes for the solution of the problem.

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